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SPECTRAL GAP ESTIMATES ON COMPACT MANIFOLDS

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ABSTRACT. For a compact Riemannian manifold with boundary, its mass gap is the difference between the first and second smallest Dirichlet eigenvalues. In this paper, taking a variational approach, we obtain an explicit lower bound estimate of the mass gap for any compact manifold in terms of geometric quantities.

0. INTRODUCTION

Let (M, g) be a compact Riemannian manifold with non-empty boundary, ∂M. The Laplace-Beltrami operator (Laplacian) acting on functions on M is defined by

$$\Delta(f) = \frac{1}{\sqrt{G}}\frac{\partial}{\partial_i}\left(\sqrt{G}g^{ij}\frac{\partial f}{\partial_j}\right),$$

where the metric on M is given by $ds^2 = g_{ij}dx^i \otimes dx^j$, $(g^{ij}) = (g_{ij})^{-1}$ and $G = det(g_{ij})$.

Eigenvalues of the Laplacian under Dirichlet boundary conditions are constants l_i which satisfy

$$\begin{cases} \Delta u_i + l_i u_i = 0, & x \in M, \\ u_i(x) = 0, & x \in \partial M, \end{cases}$$

for some nonzero eigenfunctions u_i. It is well known that the spectrum of the Dirichlet problem satisfies

$$Spec_D(M) = \{0 < l_1 < l_2 \leq l_3 \cdots \to \infty\},$$

so that the gap $l_2 - l_1$, sometimes referred to as the "mass gap" , is nontrivial.

In [S-W-Y-Y] the mass gap is estimated by analyzing the function $\phi = \frac{u_2}{u_1}$. They show that

$$\Delta\phi + 2\nabla \log u_1 \cdot \nabla \log \phi + (l_2 - l_1)\phi = 0.$$

Therefore, $l_2 - l_1$ appears as an eigenvalue of a certain partial differential operator. Moreover, since it is easy to see that ϕ is smooth on M and satisfies $\frac{\partial\phi}{\partial\nu} = 0$, $l_2 - l_1$ is a Neumann eigenvalue of a certain partial differential operator. Thus, the gradient estimate techniques of Li and Yau [L-Y] can be employed to show that

$$l_2 - l_1 \geq \frac{\pi}{4D^2}$$

where M is a convex Euclidean domain and D is the diameter of M. The assumption of convexity seems to be crucial, however, to the argument since the Hessian

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of $\log(u_1)$ appears in the gradient estimate calculation. To control this term a log-concavity result of Brascamp and Lieb [B-L] is used. The result fails for non-convex domains and manifolds.

The first author and Cheng [C-O] were able to estimate the mass gap on Euclidean domains satisfying a rolling R-ball condition,

$$l_2 - l_1 \geq C(R, H, V, n),$$

where the second fundamental form of the boundary, Π, satisfies $\Pi \geq -H$ $(H \geq 0)$, V is the volume of M and n is the dimension. This was done by introducing a weighted Cheeger's constant

$$h_u = \inf \frac{\int_H u}{\min\{\int_{M_1} u, \int_{M_2} u\}}$$

where the infimum is taken over all hypersurfaces H with $M \setminus H = M_1 \cup M_2$ and $\partial M_1 \cap \partial M_2 = H$. They then show that

$$l_2 - l_1 \geq \frac{1}{4} h_u^2$$

and give a lower bound for h_u in terms of R, H, V and n (see [C-O] for details).

In this paper, we show (Theorem 1.3)

$$l_2 - l_1 \geq C(K, H, R, D),$$

on a compact Riemannian manifold with nonempty boundary satisfying a rolling R-ball condition. In this formulation, $-K$ bounds the Ricci curvature from below, H is a positive, semi-definite matrix which bounds the second fundamental form of ∂M, R is determined by the rolling ball condition and D is the diameter of M.

As in [C-O], the key to the estimation technique is finding a lower bound for a ϕ^2 weighted Rayleigh-Ritz quotient on M, with some appropriately chosen ϕ (see Proposition 1.1, below). For suitable ϕ, for example the first Dirichlet eigenfunction, and bounded Euclidean domains satisfying rolling R-ball conditions, one can control the global behavior of the weighting function in terms of the distance function to the boundary—a Harnack-type inequality. Replacing the weighting function with the distance function allows one to use simple calculus to estimate the Rayleigh-Ritz quotient on M. Though these techniques do not carry over directly to the manifold setting, further study of the argument in [C-O] reveals that the essential elements of the estimation process are a Harnack inequality for the weighting function of the form $\phi(x) \leq C\phi(y)$ for all $x, y \in M$ satisfying $0 < d(x, \partial M) < 2d(y, \partial M)$, where C is constant and the use of a weak Neumann-Poincaré inequality on the manifold (see assumptions 1–4, below). The rolling ball condition leads to the Harnack inequality by controlling the growth of ϕ near the boundary of M in terms of the boundary geometry (H) and the global geometry (K). This naturally leads to an estimate of the constant, C, in terms of a possibly very small R, dependent on the smallest focal length as detailed in the remark following the proof of Theorem 1.3 and derived explicitly in §2.

The organization of this paper is as follows: In §1 we introduce notation and basic definitions and then prove the main theorem. In §2, which can be read independently, we prove a comparison result for the first Dirichlet eigenfunction. This verifies assumption (4), which is necesssary for the main theorem.

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1.

Let l_1 and l_2 be the first and second Dirichlet eigenvalues of M. We start with the following proposition which was established by S. Y. Cheng and the first author in [C-O].

Proposition 1.1 (Cheng & Oden)**.**

$$l_2 - l_1 = \inf_{f \in C^1(\bar{M})} \frac{\int_M |\nabla f|^2 u^2 dx}{\inf_{k\in\mathbb{R}} \int_M |f-k|^2 u^2 dx}$$

where u is the positive eigenfunction corresponding to l_1, i.e. $\Delta u - l_1 u = 0$ and $u > 0$ on M.

Proof.

$$\begin{aligned}
&\inf_{f \in C^1(\bar{M})} \frac{\int_M |\nabla f|^2 u^2}{\inf_{k\in\mathbb{R}} \int_M |f-k|^2 u^2} \\
&\quad = \inf_{\substack{f\in C^1(\bar{M}) \\ \int_M fu^2 = 0}} \frac{\int_M |\nabla f|^2 u^2}{\int_M f^2 u^2} \\
&\quad = \inf_{\substack{f\in C^1(\bar{M}) \\ \int_M fu^2 = 0}} \frac{\int_M (|\nabla (fu)|^2 - f^2 |\nabla u|^2 - 2fu\langle \nabla f, \nabla u\rangle}{\int_M (fu)^2} \\
&\quad = \inf_{\substack{f\in C^1(\bar{M}) \\ \int_M fu^2 = 0}} \frac{\int_M |\nabla (fu)|^2 - \int_M f^2 |\nabla u|^2 + \frac{1}{2}\int_M f^2 \Delta(u^2)}{\int_M (fu)^2} \\
&\quad = \inf_{\substack{f\in C^1(\bar{M}) \\ \int_M fu^2 = 0}} \frac{\int_M |\nabla (fu)|^2 - l_1 \int_M f^2 u^2}{\int_M (fu)^2} \\
&\quad = \inf_{\substack{f\in C^1(\bar{M}) \\ \int_M fu^2 = 0}} \frac{\int_M |\nabla (fu)|^2}{\int_M (fu)^2} - l_1 \\
&\quad \geq \inf_{\substack{g\in C_0^1(\bar{M}) \\ \int_M gu = 0}} \frac{\int_M |\nabla g|^2}{\int_M g^2} - l_1 \geq l_2 - l_1.
\end{aligned}$$

On the other hand, let v be an eigenfunction of l_2. Then $f_0 = \dfrac{v}{u} \in C^\infty(\bar{M})$ as shown in [S-W-Y-Y] and

$$l_2 - l_1 = \frac{\int_M |\nabla f_0|^2 u^2}{\inf_{k\in\mathbb{R}} \int_M |f_0 - k|^2 u^2} \geq \inf_{f\in C^1(\bar{M})} \frac{|\nabla f|^2 u^2}{\inf_{k\in\mathbb{R}} \int_M |f-k|^2 u^2}.$$

Thus, the proposition is proved.

In the following, we are going to use the variational characterization of $l_2 - l_1$ to give a lower bound estimate. We consider more generally the problem of estimating

$$\inf_{f\in C^1(\bar{M})} \frac{\int_M |\nabla f|^2 \varphi^2 dx}{\inf_{k\in\mathbb{R}} \int_M |f-k|^2 \varphi^2 dx},$$

where $\varphi(x)$ is a given function on M and $\varphi > 0$ on M. We assume

1. M satisfies volume doubling property, i.e. $\exists\, C_1 > 0$ s.t. for any ball $B_x(2r) \subset M$ we have $\dfrac{|B_x(2r)|}{|B_x(r)|} \le C_1$.
2. M satisfies interior rolling R-ball condition. i.e. for all $x \in \partial M$, $\exists\, B_p(R) \subset M$ s.t. $B_p(R) \cap \partial M = \{x\}$.
3. M satisfies a weak Neumann-Poincaré inequality on balls, i.e. for any ball $B_x(r) \subset M$, $B_x(r) \cap \partial M = \varnothing$, we have
$$\inf_{k\in\mathbb{R}} \int_{B_x(\frac{r}{2})} |f-k|^2 dx \le C_2 r^2 \int_{B_x(r)} |\nabla f|^2 dx$$
for all $f \in C^1(\bar{M})$, where C_2 is a constant independent of x and r.
4. $\exists\, C_3 > 0$ s.t. $\varphi(x) \le C_3 \phi(y)$ for all $x, y \in M$ with $0 < d(x, \partial M) \le 2d(y, \partial M)$.

Theorem 1.2. *Under assumptions 1, 2, 3, and 4, we have*
$$\inf_{f\in C^1(\bar{M})} \frac{\int_M |\nabla f|^2 \varphi^2 dx}{\inf_{k\in\mathbb{R}} \int_M |f-k|^2 \varphi^2 dx} \ge C(C_1, C_2, C_3, R, \mu, D),$$
where μ is the first nonzero Neumann eigenvalue of $M_{\frac{R}{2}} = \{x \in M : d(x, \partial M) \ge \dfrac{R}{2}\}$ and $D = diam(M)$, the diameter of M.

Our proof of the theorem is adapted from [J] and [SC-S].

Proof. We begin with the following claim.

Claim 1. There exists a collection of balls $\mathcal{F} = \{B_{x_i}(r_i) : i \in I\}$ s.t.

(i) $\bigcup_{i\in I} B_{x_i}(2r_i) = M$ and $B_{x_i}(r_i) \cap B_{x_j}(r_j) = \varnothing$ for any $i \ne j$, $i, j \in I$,
(ii) $d(B_{x_i}(r_i), \partial M) = 10^3 r_i$,
(iii) $\#\{B_{x_i}(r_i) \in \mathcal{F} : \bigcap B_{x_i}(10r_i) \ne \varnothing\} \le C(C_1)$.

In fact, let B_1 be a ball in M of largest radius r_1 satisfying $d(B_1, \partial M) = 10^3 r_1$. Let B_2 be a ball in M of largest radius r_2 satisfying $d(B_2, \partial M) = 10^3 r_2$ and $B_2 \cap B_1 = \varnothing$. Keep choosing balls in this way to obtain $\mathcal{F}$. We check that $\mathcal{F}$ satisfies (i), (ii) and (iii). Let $x \in M$. Then $\exists\, r > 0$ s.t. $d(B_x(r), \partial M) = 10^3 r$. We may assume $B_x(r) \notin \mathcal{F}$. It is clear then $\exists\, B \in \mathcal{F}$ s.t. the radius of B satisfies $r(B) \ge r$ and $B \cap B_x(r) \ne \varnothing$.

Write $B = B_{x_i}(r_i)$. Then for $y \in B_{x_i}(r_i) \cap B_x(r)$,
$$d(x, x_i) \le d(x, y) + d(y, x_i) \le r + r_i \le 2r_i.$$
Thus $x \in B_{x_i}(2r_i)$ and $M = \bigcup_{i\in I} B_{x_i}(2r_i)$. So (i) follows. (ii) is trivially true by the construction. To check (iii), suppose $\bigcap_{i\in J} B_{x_i}(10r_i) \ne \varnothing$. Let $y \in \bigcap_{i\in J} B_{x_i}(10r_i)$. Then $d(x_i, y) \le 10r_i$. Thus for any $x \in B_{x_i}(r_i)$, $d(x, y) \le d(x, x_i) + d(x_i, y) \le 11r_i$. Hence $\bigcup_{i\in J} B_{x_i}(r_i) \subset B_y(20r)$, where $r = \max\{r_i\} = r_{i_0}$. On the other hand,
$$d(x_{i_0}, x_j) \le d(x_i, y) + d(y, x_j) \le 10r_{i_0} + 10r_j \le 20r.$$
So
$$\begin{aligned} d(x_j, \partial M) &\ge d(x_{i_0}, \partial M) - d(x_{i_0}, x_j) \\ &\ge d(B_{x_{i_0}}(r_{i_0}), \partial M) - 20r \\ &\ge 10^3 r_{i_0} - 20r > 101r. \end{aligned}$$

Hence $d(B_{x_j}(r_j), \partial M) > 100r$ and $r_j > 10^{-1}r$. In conclusion,

$$\bigcup_{i\in J} B_{x_i}(r_i) \subset B_y(20r) \subset B_{x_j}(30r) \subset B_{x_j}(300r_j)$$

Therefore

$$\sum_{i\in J} |B_{x_i}(r_i)| \le |B_y(20r)| \le \frac{\sum_{j\in J} |B_{x_j}(300r_j)|}{\#\{B_{x_i}(r_i) : \bigcap B_{x_i}(10r_i) \ne \varnothing\}}.$$

Hence

$$\#\{B_{x_i}(r_i) : \bigcap B_{x_i}(10r_i) \ne \varnothing\} \le \frac{\sum_{j\in J} |B_{x_j}(300r_j)|}{\sum_{j\in J} |B_{x_j}(r_j)|} \le C(C_1)$$

and (iii) follows.

Now let $M_R = \{x \in M; d(x, \partial M) \ge R\}$. Let $\mathcal{L} = \{B_i \in \mathcal{F} : x_i \notin M_R\}$. For $B_i \in \mathcal{L}$, let $y_i \in \partial M$ s.t. $d(x_i, y_i) = d(x_i, \partial M)$. By the interior rolling R-ball condition, $\exists\, B_{q_i}(R)$ s.t. $B_{q_i}(R) \cap \partial M = y_i$. Then clearly $d(q_i, y_i) = d(q_i, \partial M) = R$. Let $\overline{q_i y_i}$ be a minimal geodesic. Then $\overline{q_i y_i} \perp \partial M$. In particular, this implies such a minimal geodesic is unique up to a reparametrization. Also, let $\overline{x_i y_i}$ be a minimal geodesic realizing the distance between x_i and ∂M. Then $\overline{x_i y_i} \perp \partial M$. This forces $x_i \in \overline{q_i y_i}$. We denote by l_i the segment $\overline{q_i x_i}$ of $\overline{q_i y_i}$. We then define, for $B_i \in \mathcal{L}$, $\mathcal{F}(B_i) = \{A \in \mathcal{F} : 2A \cap l_i \ne \varnothing\}$. Let $\mathcal{H} = \{A \in \mathcal{F} : A \in \mathcal{F}(B_i) \text{ for some } B_i \in \mathcal{L}\}$. We have the following claim.

Claim 2. (i) $A \in \mathcal{F}(B)$ implies $r(A) \ge 10^{-1} r(B)$.

(ii) For any $A \in \mathcal{H}$, let

$$A(\mathcal{L}) = \{B \in \mathcal{L} : A \in \mathcal{F}(B)\}.$$

Then

$$r(A)^2 |A|^{-1} \sum_{B\in A(\mathcal{L})} \#\mathcal{F}(B)|B| \le C(C_1) R^2.$$

We first check (i). For $A \in \mathcal{F}(B)$, $2A \cap l_B \ne \varnothing$. Let $y \in 2A \cap l_B$ and x be the center of the ball B. Then

$$\begin{aligned} d(A, \partial M) &\ge d(y, \partial M) - d(A, y) \\ &\ge d(x, \partial M) - d(A, y) \\ &\ge 10^3 r(B) - d(A, y) \\ &\ge 10^3 r(B) - 2r(A). \end{aligned}$$

Thus $10^3 r(A) \ge 10^3 r(B) - 2r(A)$ and $r(A) \ge 10^{-1} r(B)$.

To check (ii), we show

(a) $\#\{A \in \mathcal{F}(B) : r \le r(A) \le \frac{11}{10} r\} \le C(C_1)$ for any $r > 0$, where C is independent of r.

(b) $\#\mathcal{F}(B) \le C \ln \frac{CR}{r(B)}, C = C(C_1)$.

(c) $\exists\, \varepsilon = \varepsilon(C_1) > 0$ s.t.

$$\sum_{B\in A(\mathcal{L}), r\le r(B)\le 2r} |B| \le C|A| (\frac{r}{r(A)})^{\varepsilon} \text{ for } A \in \mathcal{H}$$

where $r > 0$ is arbitrary, $C = C(C_1)$ constant.

For (a), suppose $A_1, A_2 \in \{A \in \mathcal{F}(B) : r \leq r(A) \leq \frac{11}{10}r\}$ and $y_1 \in 2A_1 \cap l_B,\ y_2 \in 2A_2 \cap l_B$. Let $A_1 = B_{\xi_1}(r_1)$ and $A_2 = B_{\xi_2}(r_2)$. Since for $i = 1,\ 2$,

$$d(y_i, \partial M) \geq d(\xi_i, \partial M) - d(\xi_i, y_i) \geq 10^3 r_i - 2r_i$$

and

$$d(y_i, \partial M) \leq d(\xi_i, \partial M) + d(\xi_i, y_i) \leq 10^3 r_i + 3r_i,$$

we have

$$\begin{aligned} d(y_1, y_2) &\leq |d(y_1, \partial M) - d(y_2, \partial M)| \\ &\leq (10^3 + 3)\frac{11}{10}r - (10^3 - 2)r \\ &\leq 15r. \end{aligned}$$

Now it is clear that

$$A_2 \subset B_{y_1}(20r) \text{ and } d(y_1, \partial M) \geq 998r.$$

Therefore

$$\bigcup\{A \in \mathcal{F}(B) : r \leq r(A) \leq \frac{11}{10}r\} \subset B_{y_1}(20r).$$

Using the volume doubling property, we conclude

$$\#\{A \in \mathcal{F}(B) : r \leq r(A) \leq \frac{11}{10}r\} \leq C(C_1).$$

Now (b) follows from (a). In fact, $A \in \mathcal{F}(B)$ implies that $10^{-1}r(B) \leq r(A) \leq R$. Therefore

$$\begin{aligned} \#\mathcal{F}(B) &\leq \#\{A \in \mathcal{F}(B) : 10^{-1}r(B) \leq r(A) \leq \frac{11}{10}10^{-1}r(B)\} \\ &+ \#\{A \in \mathcal{F}(B) : \frac{11}{10}10^{-1}r(B) \leq r(A) \leq (\frac{11}{10})^2 10^{-1}r(B)\} \\ &+ \ldots + \#\{A \in \mathcal{F}(B) : (\frac{11}{10})^k 10^{-1}r(B) \leq r(A) \leq (\frac{11}{10})^{k+1} 10^{-1}r(B)\} \end{aligned}$$

where k satisfies $(\frac{11}{10})^k 10^{-1}r(B) \leq R \leq (\frac{11}{10})^{k+1}10^{-1}r(B)$. From (a) we know $\#\mathcal{F}(B) \leq Ck$. But $k \leq C \ln \frac{CR}{r(B)}$. Thus $\#\mathcal{F}(B) \leq C \ln \frac{cR}{r(B)}$.

Now we come to (c). First, note that for $B \in A(\mathcal{L}), r(B) < 10r(A)$ by (i) of Claim 2. So $\frac{r}{r(A)} < 10$. Choose $\eta_0 \in \partial M$ s.t. $d(\eta_0, A) = d(A, \partial M) = 10^3 r(A)$. Let $s = 10^{10}r(A)$. For $\delta > 0$, denote

$$R(\delta) = \{\eta \in B(\eta_0, s + 2\delta s) : d(\eta, \partial M) < \delta s\}.$$

Then for $B \in A(\mathcal{L})$ and $r(B) \leq 2r$, we have $B \subset R(\frac{r}{r(A)})$. In fact, let $B = B_x(r(B))$. Then

$$d(x, \partial M) \leq d(B, \partial M) + r(B) \leq 10^3 r(B) + 2r \leq 10^4 r.$$

So $B \subset \{\eta : d(\eta, \partial M) < 10^{10}r\}$. To show $B \subset B(\eta_0, s + 2\delta s)$, $\delta = \frac{r}{r(A)}$, note that $A \in \mathcal{F}(B)$. Therefore, $\exists\ y \in \overline{xq} \cap 2A$. Hence

$$d(x,y) \leq d(y,\partial M) \leq 3r(A) + d(A,\partial M) \leq (10^3+3)r(A).$$

So

$$d(x,\eta_0) \leq d(x,y) + d(y,\eta_0) \leq 2(10^3+3)r(A)$$

and for $z \in B_x(r(B))$,

$$d(z,\eta_0) \leq d(z,x) + d(x,\eta_0) < 10^{10}r(A) + 2r.$$

Therefore $B_x(r(B)) \subseteq R(\dfrac{r}{r(A)})$. On the other hand, by the volume doubling property, we have $|A| \approx |B(\eta_0, 2s)|$. Thus, (c) follows if we can show

$$|R(\delta)| \leq C\delta^{\varepsilon}|B(\eta), 2s)|, \qquad 0 < \delta < 10.$$

We first show that $\exists\ \sigma(C_1) > 0$ s.t.

$$(1+\sigma)|R(\frac{1}{4}\delta)| \leq |R(\delta)| \ \text{ for } \ 0 < \delta < 10^{10}.$$

Choose a maximal set of points $\xi_1, \dots, \xi_N \in \partial M$ s.t. $B(\xi_i, \delta s) \cap B(\eta_0, s) \neq \varnothing$ and $\{B(\xi_i, \delta s);\ i = 1, \dots, N\}$ are pairwise disjoint. It is then easy to check that

$$R(\frac{1}{4}\delta) \subset \bigcup_{j=1}^{N} B(\xi_j, 10\delta s).$$

Also, $\exists\ B(\eta_j, \dfrac{\delta s}{2}) \subset \bar{M}\backslash\partial M$ s.t. $d(\eta_j, \xi_j) \leq \dfrac{\delta s}{2}$. In fact, since $A \in \mathcal{H}$, $A \in \mathcal{F}(B)$ for some $B \in \mathcal{L}$. Thus, $d(A, \partial M) \leq r(A) + R$ and $r(A) \leq 10^{-2}R$. By the interior rolling R-ball condition, $\exists\ q_j$ s.t. $B_{q_j}(R) \cap \partial M = \{\xi_j\}$. Now choose $\eta_j \in \overline{q_j\xi_j}$ s.t. $d(\eta_j, \xi_j) = \dfrac{\delta s}{2} \leq \dfrac{r(A)}{2} < 10^{-2}R$. Then

$$B(\eta_j, \frac{\delta s}{2}) \subset B_{q_j}(R) \subset \bar{M}\backslash\partial M \text{ and } d(\eta_j, \xi_j) = \frac{\delta s}{2}.$$

Now we have

$$B(\eta_j, \frac{\delta s}{4}) \subset R(\delta) \quad \text{and} \quad B(\eta_j, \frac{\delta s}{4}) \cap R(\frac{1}{4}\delta) = \varnothing.$$

Moreover, $\exists\ \sigma > 0$ s.t. $|B(\eta_j, \dfrac{\delta s}{4})| > \sigma|B(\xi_j, 10\delta s)|$. Therefore,

$$\sum_{j=1}^{N} |B(\eta_j, \frac{\delta s}{4})| > \sigma \sum_{j=1}^{N} |B(\xi_j, 10\delta s)| \geq \sigma|R(\frac{1}{4}\delta)|.$$

Consequently,

$$|R(\delta)| \geq |R(\frac{1}{4}\delta)| + |\bigcup_{j=1}^{N} B(\eta_j, \frac{\delta s}{4})| > (1+\sigma)|R(\frac{1}{4}\delta)|$$

as the balls $B(\eta_j, \dfrac{\delta s}{4})$ are mutually disjoint.

For any $\delta > 0$, write $\delta \cong (\frac{1}{4})^k \delta_0$, where δ_0 fixed. Then

$$|R(\delta)| \leq (\frac{1}{1+\sigma})^k |R(\delta_0)| \leq C\delta^\varepsilon |B(\eta_0, 2s)|.$$

Thus (c) follows.

Now (ii) of Claim 2 can be checked. In fact,

$$\begin{aligned}
& r(A)^2|A|^{-1} \sum_{B\in A(\mathcal{L})} \#\mathcal{F}(B)|B| \\
\leq\ & r(A)^2|A|^{-1} \sum_{B\in A(\mathcal{L})} C\ln\frac{CR}{r(B)}|B| \\
\leq\ & r(A)^2|A|^{-1} \sum_{k=-\infty}^{0} \sum_{\substack{B\in A(\mathcal{L}) \\ 2^k r(A)\leq r(B)\leq 2^{k+1} r(A)}} C\ln\frac{CR}{r(B)}|B| \\
\leq\ & r(A)^2|A|^{-1} \sum_{k=-\infty}^{0} C\ln\frac{CR}{2^k r(A)}|A|(\frac{2^k r(A)}{r(A)})^\varepsilon \\
\leq\ & r(A)^2 \sum_{k=-\infty}^{0} [(C\ln\frac{CR}{r(A)})2^{k\varepsilon} - (C\ln 2)k2^{k\varepsilon}] \\
\leq\ & r(A)^2[C\ln\frac{CR}{r(A)} + C] \\
\leq\ & CR^2 \quad \text{as} \quad r(A) \leq CR.
\end{aligned}$$

With those two claims, we can now finish the proof of the theorem.

For $B \in \mathcal{L}$, let $A_1, \ldots, A_l$ be the elements in $\mathcal{F}(B)$ s.t. $A_1 = B$, $q \in 2A_l$. Let $f'_B = \int_B \frac{f\varphi^2}{\int_B \varphi^2}$, $f'_0 = \frac{\int_{M_{R/2}} f\varphi^2}{\int_{M_{R/2}} \varphi^2}$. Then

$$\begin{aligned}
& \int_{4B} |f - f'_0|^2\varphi^2 \\
\leq\ & 2\int_{4B} |f - f'_{4B} + \sum_{i=1}^{l-2}(f'_{4A_i} - f'_{4A_{i+1}})|^2\varphi^2 + 2\int_{4B}\varphi^2|f'_{l-1} - f'_0|^2 \\
\leq\ & 2l[\int_{4B}|f - f'_{4B}|^2\varphi^2 + \int_{4B}\varphi^2\sum_{i=1}^{l-2}|f'_{4A_i} - f'_{4A_{i+1}}|^2] + 2\int_{4B}\varphi^2|f'_{l-1} - f'_0|^2.
\end{aligned}$$

Since

$$\begin{aligned}
& \int_{4A_i\cap 4A_{i+1}} \varphi^2|f'_{4A_i} - f'_{4A_{i+1}}|^2 \\
\leq\ & 2[\int_{4A_i}|f - f'_{4A_i}|^2\varphi^2 + \int_{4A_{i+1}}|f - f'_{4A_{i+1}}|^2\varphi^2],
\end{aligned}$$

and

$$\begin{aligned}
&\int_{4B} \varphi^2 |f'_{l-1} - f'_0|^2 \\
\leq\ & \frac{\int_{4B} \varphi^2}{\int_{4A_{l-1}\cap M_{R/2}} \varphi^2} \Big(\int_{4A_{l-1}\cap M_{R/2}} \varphi^2 |f'_{l-1} - f'_0|^2\Big) \\
\leq\ & 2\frac{\int_{4B} \varphi^2}{\int_{4A_{l-1}\cap M_{R/2}} \varphi^2} \Big(\int_{4A_{l-1}\cap M_{R/2}} (|f - f'_{l-1}|^2 + |f - f'_0|^2)\varphi^2\Big) \\
\leq\ & 2\frac{\int_{4B} \varphi^2}{\int_{4A_{l-1}\cap M_{R/2}} \varphi^2} \Big(\int_{4A_{l-1}} |f - f'_{l-1}|^2\varphi^2 + \int_{M_{R/2}} |f - f'_0|^2\varphi^2\Big),
\end{aligned}$$

we conclude

$$\begin{aligned}
&\int_{4B} |f - f'_0|^2\varphi^2 \\
\leq\ & 2l\Big[\int_{4B} |f - f'_{4B}|^2\varphi^2 + \sum_{i=1}^{l-2} \frac{\int_{4B} \varphi^2}{\int_{4A_i\cap 4A_{i+1}} \varphi^2} \int_{4A_i\cap 4A_{i+1}} \varphi^2 |f'_{4A_i} - f'_{4A_{i+1}}|^2\Big] \\
& +4\frac{\int_{4B} \varphi^2}{\int_{4A_{l-1}\cap M_{R/2}} \varphi^2} \Big(\int_{4A_{l-1}} |f - f'_{l-1}|^2\varphi^2 + \int_{M_{R/2}} |f - f'_0|^2\varphi^2\Big) \\
\leq\ & 4l\Big[\int_{4B} |f - f'_{4B}|^2\varphi^2 \\
& \quad + \sum_{i=1}^{l-2} \frac{\int_{4B} \varphi^2}{\int_{4A_i\cap 4A_{i+1}} \varphi^2} \Big(\int_{4A_i} |f - f'_{4A_i}|^2\varphi^2 + \int_{4A_{i+1}} |f - f'_{4A_{i+1}}|^2\varphi^2\Big)\Big] \\
& +4\frac{\int_{4B} \varphi^2}{\int_{4A_{l-1}\cap M_{R/2}} \varphi^2} \Big(\int_{4A_{l-1}} |f - f'_{l-1}|^2\varphi^2 + \int_{M_{R/2}} |f - f'_0|^2\varphi^2\Big).
\end{aligned}$$

From assumption (4) on φ, it is easy to see that

$$\frac{\int_{4B} \varphi^2}{\int_{4A_i\cap 4A_{i+1}} \varphi^2} \leq C(C_3)\frac{|B|}{|4A_i \cap 4A_{i+1}|} \leq C(C_1, C_3)\frac{|B|}{|A_i|}$$

and

$$\frac{\int_{4B} \varphi^2}{\int_{4A_{l-1}\cap M_{R/2}} \varphi^2} \leq C(C_1, C_3)\frac{|B|}{|A_{l-1}|}.$$

On each A_i, we also have $\varphi(x) \leq C_2\varphi(y)$ for all $x, y \in A_i$. Therefore by assumption (3),

$$\int_{4A_i} |f - f'_i|^2\varphi^2 \leq C_2 r^2(A_i) \int_{4A_i} |\nabla f|^2\varphi^2.$$

On $M_{R/2}$, $\varphi(x) \leq C_4(C_3, R, D)\varphi(y)$ for $x, y \in M_{R/2}$, where D is the diameter of M. Thus

$$\int_{M_{R/2}} |f - f'_0|^2\varphi^2 \leq \frac{C_4}{\mu} \int_{M_{R/2}} |\nabla f|^2\varphi^2.$$

In conclusion, we have

$$\begin{aligned}\int_{4B}|f-f_0'|^2\varphi^2 &\leq C\#\mathcal{F}(B)|B|\sum_{A_i\in\mathcal{F}(B)} r(A_i)^2|A_i|^{-1}\int_{4A_i}|\nabla f|^2\varphi^2\\ &\quad+\frac{C_4|B|}{\mu|A_l|}\int_{M_{R/2}}|\nabla f|^2\varphi^2.\end{aligned}$$

Using assumption (1), it is easy to see $|A_l|\geq C(C_1,D,R)|M|$. Thus

$$\begin{aligned}\int_{4B}|f-f_0'|^2\varphi^2 &\leq C\#\mathcal{F}(B)|B|\sum_{A_i\in\mathcal{F}(B)} r(A_i)^2|A_i|^{-1}\int_{4A_i}|\nabla f|^2\varphi^2\\ &\quad+\frac{C_5|B|}{\mu|M|}\int_{M_{R/2}}|\nabla f|^2\varphi^2.\end{aligned}$$

Summing over all $B\in\mathcal{L}$, we have

$$\begin{aligned}&\int_{M\setminus M_R}|f-f_0'|^2\varphi^2\\ \leq\ &\sum_{B\in\mathcal{L}}\int_{4B}|f-f_0'|^2\varphi^2\\ \leq\ &C\sum_{B\in\mathcal{L}}\#\mathcal{F}(B)|B|\sum_{A\in\mathcal{F}(B)} r(A)^2|A|^{-1}\int_{4A}|\nabla f|^2\varphi^2+C_5\frac{|M\setminus M_R|}{\mu|M|}\int_{M_{R/2}}|\nabla f|^2\varphi^2\\ \leq\ &C\sum_{A\in\mathcal{H}} r(A)^2|A|^{-1}\sum_{B\in A(\mathcal{L})}|B|\#\mathcal{F}(B)\int_{4A}|\nabla f|^2\varphi^2+\frac{C_5}{\mu}\int_{M_{R/2}}|\nabla f|^2\varphi^2\\ \leq\ &CR^2\int_M|\nabla f|^2\varphi^2+\frac{C_5}{\mu}\int_{M_{R/2}}|\nabla f|^2\varphi^2,\end{aligned}$$

where we have used (ii) of Claim 2. Thus

$$\begin{aligned}\int_M|f-f_0'|^2\varphi^2 &\leq \int_{M\setminus M_R}|f-f_0'|^2\varphi^2+\int_{M_{R/2}}|f-f_0'|^2\varphi^2\\ &\leq (CR^2+\frac{C_5+1}{\mu})\int_M|\nabla f|^2\varphi^2.\end{aligned}$$

Therefore

$$\frac{\int_M|\nabla f|^2\varphi^2}{\inf_{k\in\mathbb{R}}\int_M|f-k|^2\varphi^2}\geq\frac{1}{CR^2+\frac{C_6}{\mu}},$$

where $C_6=C(C_1,C_2,C_3,R,D)$ and $C=C(C_1,C_2,C_3)$. The theorem is proved.

Remark. Notice that the assumptions (1), (2) and (3) are stable under quasi-isometry on (M,g).

In order to apply Theorem 1.1 to estimate l_2-l_1, we need to verify assumptions (1), (2) and (3) for (M,g) and (4) for u. (4) is verified in §2. To verify assumptions (1), (2) and (3) we assume now the following for (M,g).

(5) $Ric_M\geq -K$ on M for some constant $K\geq 0$.

(6) The second fundamental form of ∂M w.r.t. the outward unit normal satisfies $\Pi \geq -H$, where $H \geq 0$.

Theorem 1.3. *Suppose (M, g) satisfies assumptions (2), (5) and (6). Then $l_2 - l_1 \geq C(K, H, R, D)$, where R is chosen to be "small". (See Remark.)*

Proof. It has been shown in [W] that M satisfies (1) with $C_1 = C(K, R)$ by arranging R "small". M satisfies (3) by fact $Ric_M \geq -K$ and $C_2 = C(K, D)$ (see [SC-S]). To estimate μ from below, we use the result in [C].

Note first that $M_{R/2}$ satisfies interior r-rolling ball condition with $r \geq \dfrac{R}{4}$. In fact, $\forall\ x \in \partial M_{R/2}$, $\exists\ y \in \partial M$ s.t. $d(x, y) = d(x, \partial M)$. Since M satisfies interior R-rolling ball condition, $\exists\ B_p(R) \subset M$ s.t. $B_p(R) \cap \partial M = \{y\}$. Now let $\overline{py}$ be the minimal geodesic. Let $q \in \overline{py}$ s.t. $d(p, q) = \overline{pq} = \dfrac{1}{4}R$. Then it can be easily checked that $B_q(\dfrac{1}{4}R) \subset M_{R/2}$ and $B_q(\dfrac{1}{4}R) \cap \partial M_{R/2} = \{x\}$.

Next, we want to estimate the second fundamental form of $\partial M_{R/2}$ from below. Let $f(x) = d(x, \partial M)$. Then $M_{R/2} = \{f(x) \geq \dfrac{R}{2}\}$ and $\partial M_{R/2} = \{f(x) = \dfrac{R}{2}\}$. Since $|\nabla f| = 1$ on $M \backslash M_R$, it is easy to see that $\Pi_{\partial M_{R/2}} = Hessian(f)$ on M. By the index comparison theorem in [H-K] or [Wr], we have, by choosing R small,

$$Hessian(f) \geq -C(R, H).$$

Thus, $\Pi_{\partial M_{R/2}} \geq -C(R, H)$. By [C], we conclude that

$$\mu \geq C(R, H, K, D)$$

In conclusion, we have

$$l_2 - l_1 \geq C(R, D, K, H)$$

where R is chosen to be "small".

Remark. The number R is chosen to satisfy the following:

(a) $\sqrt{K_R} \tan(R\sqrt{K_R}) \leq \frac{H}{2} + \frac{1}{2}$,

(b) $H \tan(R\sqrt{K_R}) \leq \frac{1}{2}\sqrt{K_R}$, where K_R is the upper bound of the sectional curvature of M on the set $M \setminus M_R$. Clearly, if M is a Euclidean domain, then $K_R = 0$ and R can be taken as in (2).

2.

In this section we will verify property (4) for the first eigenfunction given Dirichlet conditions on a manifold M with $\partial M \neq \emptyset$.

In order to do this we will need bounds on $\Delta d(x, \partial M)$ for x "close " to ∂M. To this end we recall the notation from §1 and set

$$M_\delta = \{x \in M | d(x, \partial M) > \delta\}.$$

We define a map $\Psi : \partial M \times [0, \infty) \longrightarrow M$ by setting $\Psi(y, \rho) = \exp(\rho N_y)$ where N_y is the inward pointing unit normal at $y \in \partial M$ and recall the following standard facts about Ψ:

1. If ∂M satisfies an interior rolling R-ball condition, then there exists $\delta > 0$ such that $\Psi : \partial M \times [0, \delta) \longrightarrow M$ is C^2.
2. For the δ of (1), $d(x, \partial M) = d(x, y) = \rho$, and $\rho(x)$ is C^2 for $x \in M \setminus M_\delta$.

3. If dV is the volume element on M and Φ is the volume element on ∂M, then $\Psi^*(dV) = \mathcal{H}(y,\rho)\Phi \wedge d\rho$ and $\Delta\rho(x) = \Delta d(x,\partial M) = \frac{\mathcal{H}'}{\mathcal{H}}$ on $M \setminus M_\delta$.

Many proofs of properties (1), (2) and (3) can be found in the literature. A proof of property (3) can be found in [G-H-L]. Proofs of (1),(2) and (3) can also be found in [O]. In the next lemma we shall bound $\Delta d(x,\partial M)$ as well as get an estimate on δ.

Lemma 2.1. *Let $\rho(x) = d(x,\partial M)$. Suppose we have constants $b \geq a > 0$, $b \geq 1$ such that $-a^2 \leq Sec(M) \leq b^2$. Then ρ is C^k on $M \setminus M_\delta$, if ∂M is C^k and*

$$\sum_{i=1}^{n-1} \frac{-b\sin b\rho - k_i(y)\cos b\rho}{\cos b\rho - \frac{k_i(y)}{b}\sin b\rho} \leq \Delta\rho \leq \sum_{i=1}^{n-1} \frac{-a\sinh a\rho - k_i(y)\cosh a\rho}{\cos ha\rho - \frac{k_i(y)}{a}\sinh a\rho}$$

where $k_i(y)$ are the principal curvatures on ∂M at the unique point $y \in \partial M$ such that $\rho(x) = d(x,y)$ and

$$\delta = \frac{1}{4(n-1)(b + H + 1/R)}.$$

Proof. Let $x \in M$ with $\rho(x) = d(x,\partial M) = d(x,y)$ with $y \in \partial M$. Suppose k_i are the principal curvatures at y. At least locally, on the space form of constant curvature b^2 we may immerse a hypersurface with curvatures k_i. To derive the first inequality, we consider a hypersurface $\overline{H}$ immersed in $S^n(1/b)$, the n-sphere of radius $1/b$. The second inequality is derived by replacing $S^n(1/b)$ with the hyperbolic space $H^n(1/a)$ and repeating the argument. Let $D \subseteq \overline{H}$ be an open set and define, as above, $\overline{\Psi} : D \times [0,\infty) \to S^n(1/b)$ by

$$\overline{\Psi}(y,\overline{\rho}) = \exp \overline{\rho} N_y$$

where N_y is the inward pointing unit normal at y. Since geodesics on $S^n(1/b)$ starting at a point y and having initial direction N_y have the form

$$\cos b\overline{\rho} y + \frac{1}{b}\sin b\overline{\rho} N_y$$

we may write

$$\overline{\Psi}(y,\overline{\rho}) = \cos b\overline{\rho} y + \frac{1}{b}\sin b\overline{\rho} N_y.$$

On $\overline{H}$ there exist $n-1$ orthonormal principal vectors E_i. We extend E_i by parallel translation to vector fields $E_i(\overline{\rho})$ along the normal geodesic $\gamma(\overline{\rho}) = \overline{\Psi}(y,\overline{\rho})$. Let $\gamma_i(t)$ be geodesics on $\overline{H}$ such that $\gamma_i(0) = y$ and $\gamma_i'(0) = E_i(0)$. Then

$$\begin{aligned} \frac{\partial \overline{\Psi}}{\partial \gamma_i}(y,\overline{\rho}) &= \cos b\overline{\rho}\gamma_i'(0) + \frac{1}{b}\sin b\overline{\rho}\frac{\partial N_{\gamma_i(t)}}{\partial \gamma_i}(0) \\ &= \cos b\overline{\rho} E_i - \frac{k_i}{b}\sin b\overline{\rho} E_i \\ &= \left(\cos b\overline{\rho} - \frac{\overline{k_i}}{b}\sin b\overline{\rho}\right) E_i. \end{aligned}$$

Since $\left(\cos b\overline{\rho} - \frac{k_i}{b}\sin b\overline{\rho}\right) E_i$ are variation vector fields, they are in fact Jacobi fields and describe the Jacobian, $J(\overline{\Psi})$, of $\overline{\Psi}$ (see for instance [H-K]), and $J(\overline{\Psi})$ satisfies

$$\begin{aligned} J(\overline{\Psi})(y,\bar{\rho}) &= (\cos b\overline{\rho} - \frac{k_1}{b}\sin b\overline{\rho})(\cos b\overline{\rho} - \frac{\overline{k_2}}{b}\sin b\overline{\rho})\cdots(\cos b\overline{\rho} - \frac{k_{n-1}}{b}\sin b\overline{\rho}) \\ &> 0. \end{aligned}$$

The last inequality follows from $\overline{\rho} \leq \delta$ so that

$$\cot b\overline{\rho} \geq \frac{2H}{b}.$$

We have used the fact that $\theta \longrightarrow 0,\ \tan\theta$ is approximately θ. Therefore, we may apply the inverse function thereom to find an open set $D' \subseteq D$ where y is a C^{k-1} function of $x = \overline{\Psi}(y, \overline{\rho})$. However, the definition of $\overline{\Psi}$ implies $\nabla\overline{\rho} = N_y$ is a C^{k-1} function of y. Therefore, $\overline{\rho}$ is C^k. The index estimates on $\overline{\mathcal{H}}$ and $\mathcal{H}$ in [H-K] imply $J(\overline{\Psi})(y, \overline{\rho}) = \overline{\mathcal{H}}(y, \overline{\rho}) \leq \mathcal{H}(y, \rho)$ where $\overline{\rho} = \rho$ which implies ρ is C^k on $M \setminus M_\delta$. Applying property (2) we have

$$\Delta\rho \geq \Delta\overline{\rho} = \sum_{i=1}^{n-1} \frac{-b\sin b\rho - k_i(y)\cos b\rho}{\cos b\rho - \frac{k_i(y)}{b}\sin b\rho},$$

which completes the proof of the lemma.

Since $\rho \leq \delta$ from Lemma 2.1 we can draw the following conclusions:

$$\begin{aligned} \Delta\rho &\geq \sum_{i=1}^{n-1} \frac{-b\sin b\rho - k_i(y)\cos b\rho}{\cos b\rho - \frac{k_i(y)}{b}\sin b\rho} \\ &= \sum_{i=1}^{n-1} \frac{-b\tan b\rho - k_i(y)}{1 - \frac{k_i(y)}{b}\tan b\rho} \\ &\geq \sum_{i=1}^{n-1} -2\,(b + k_i) \\ &\geq -2(n-1)(b + H + 1/R). \end{aligned} \tag{2.1}$$

$$\begin{aligned} \Delta\rho^2 &= 2\rho\Delta\rho + 2|\nabla\rho|^2 \\ &\geq 2\rho\left(-2(n-1)(b + H + 1/R)\right) + 2 \\ &\geq -1 + 2 \\ &\geq 1. \end{aligned} \tag{2.2}$$

Now we can place $\sup_M u$ on a compact subset of M:

Lemma 2.2. *Suppose u is the first eigenfunction on M and $u(x_0) = \sup_{x\in M} u$. Then on $M \setminus M_\delta$*

$$u(x) \leq \frac{6\mu}{\delta}\rho(x).$$

Furthermore,

$$d(x_0, \partial M) \geq \frac{\delta}{6\mu}$$

where

$$\mu = \frac{(n-1)a^2}{4} + \frac{4\pi^2}{\delta^2} + \frac{(n-1)^2a^2}{4\sinh^2(a/2)}.$$

Proof. Without loss of generality, assume $\sup_{x\in M} u = 1$. Set $\gamma_1 = 2/(3\delta)$. Then using Lemma 2.1 and conclusions (2.1) and (2.2) we have

$$\begin{aligned}\Delta\left(\rho-\gamma_1\rho^2\right) &\le \sum_{i=1}^{n-1}\frac{-a\tanh a\rho-k_i}{1-\frac{k_i}{a}\tanh a\rho}-\gamma_1\\ &\le 2(n-1)(a+H+1/R)-\frac{1}{2\delta}\\ &\le \frac{1}{2\delta}-\frac{2}{3\delta}\\ &= -\frac{1}{6\delta}.\end{aligned}$$

A result of Gage ([G]) tells us that $\lambda \le \mu$, so by setting $\gamma_2 = 6\mu/\delta$ we have

$$\begin{aligned}\Delta\left(u-\gamma_2(\rho-\gamma_1\rho^2)\right) &\ge -\lambda u+\gamma_2\frac{1}{6\delta}\\ &\ge -\lambda u+\mu\\ &\ge 0.\end{aligned}$$

On ∂M_δ we have

$$\begin{aligned}u-\gamma_2(\rho-\gamma_1\rho^2) &= u-\gamma_2(\delta-\gamma_1\delta^2)\\ &= u-\gamma_2\left(\delta-\frac{1}{2}\delta\right)\\ &= u-\frac{6\mu}{\delta}\frac{1}{2}\delta\\ &\le u-3\mu\\ &\le 0.\end{aligned}$$

The last inequality follows since μ is clearly greater than 1. Therefore, by the maximum principle we have

$$d(x,\partial M)=\rho(x)\ge\frac{1}{\gamma_2}u(x).$$

To bound u from below is more involved. Again the maximum principle will be used but it is necessary for us to bound u from below on $\{\rho=\epsilon\}$ for some suitable ϵ. This is done in the following proposition.

Proposition 2.3. *Let u be the positive eigenfunction of M with respect to Dirichlet boundary conditions and normalized so that $\sup_M u = 1$. Then, for $d(x,\partial M)\ge\frac{\delta}{6\mu}$ we have*

$$u(x)\ge\exp\left(-\frac{24\mu d_M}{\delta}\log\left(\frac{50(n-1)^2}{\delta}\left(\mu^2+a\right)\right)\right). \tag{2.3}$$

Proof. Proposition says that $\sup_{M_\epsilon} u = 1$ with $\epsilon=\frac{6\mu}{\delta}$. We will apply the Harnack inequality to bound u from below on M_ϵ. For any $x\in M_\epsilon$, $B_\epsilon(x)\subseteq M$. The gradient estimate in [S-Y] says that

$$\left|\frac{\nabla u}{u}\right|\le 4\frac{l_1^2}{\epsilon^2}+\frac{8(n-1)^2}{\epsilon^2}\left(\epsilon^2+a\right)$$

on $B_{\epsilon/2}(x)$. Therefore, on $B_{\epsilon/2}(x)$,

$$\sup_{B_{\epsilon/2}(x)} u \leq \frac{l_1^2}{\epsilon} + \frac{8(n-1)^2}{\epsilon^2}\,(\epsilon + a) \inf_{B_{\epsilon/2}(x)} u.$$

Since $l_1 \leq \mu$, we have

$$\sup_{B_{\epsilon/2}(x)} u \leq \frac{8(n-1)^2}{\epsilon}\left(\mu^2 + a\right) \inf_{B_{\epsilon/2}(x)} u. \tag{2.4}$$

Since $\epsilon = \frac{6\mu}{\delta} \leq \frac{R}{2}$ for $n \geq 2$, we see easily that $d_{M_\epsilon} \leq 2d_M$. Applying (2.4) repeatedly on M_ϵ we derive that

$$1 = \sup_{M_\epsilon} u \leq \left(\frac{8(n-1)^2}{\epsilon}\left(\mu^2 + a\right)\right)^{\frac{4d_M}{\epsilon}} \inf_{M_\epsilon} u.$$

Therefore,

$$\inf_{M_\epsilon} u \geq \exp\left(-\frac{24\mu d_M}{\delta}\log\left(\frac{50(n-1)^2}{\delta}\left(\mu^2 + a\right)\right)\right).$$

This completes the proof of the proposition.

Now we can bound u from below by some multiple of ρ.

Proposition 2.4. *Let u be the first positive eigenfunction of M with respect to Dirichlet boundary conditions and normalized so that $\sup_M u = 1$. For $x \in M$, $d(x, \partial M) \leq \frac{6\mu}{\delta}$, we have*

$$u(x) \geq \rho(x)\frac{1}{4\delta}\exp\left(-\frac{24\mu d_M}{\delta}\log\left(\frac{50(n-1)^2}{\delta}\left(\mu^2 + a\right)\right)\right).$$

Proof. Let $C(\delta)$ denote the right-hand side of (2.3).

Let α_1, α_2 be positive constants to be chosen later. Then on T_δ,

$$\begin{aligned}\Delta\left(u - \alpha_2(\rho + \alpha_1\rho^2)\right) &= -l_1 u - \alpha_2\left(\sum \frac{-b\sinh b\rho - k_1\cosh b\rho d}{\cos b\rho - \frac{k_1}{b}\sin b\rho} + \alpha_1\Delta\rho^2\right)\\ &\leq -l_1 u - \alpha_2\left(-2(n-1)(b + H + 1/R) + \alpha_1\right). \qquad (2.5)\end{aligned}$$

Therefore, the right-hand side of (2.5) will be less than or equal to zero if $\alpha_1 \geq (n-1)(b + H + 1/R)$.

Set $\alpha_1 = \frac{1}{\delta}$. Then, the infimum of $\left(u - \alpha_2(\rho + \alpha_1\rho^2)\right)$ on $M \setminus M_\delta$ occurs on the boundary of $M \setminus M_\delta$.

For $x \in M$, $d(x, \partial M) = \delta$, we have

$$\begin{aligned}u(x) - \alpha_2\left(\delta + \alpha_1\delta^2\right) &\geq C(\delta) - \alpha_2\delta\left(1 + \frac{1}{\delta}\delta\right)\\ &\geq C(\delta) - 2\alpha_2\delta\\ &\geq 0,\end{aligned}$$

if $\alpha_2 = \frac{C(\delta)}{4\delta}$.

So, with these choices of α_1, α_2 we then have on $M \setminus M_\delta$,

$$u \geq \frac{C(\delta)}{4\delta}\left(\rho + \frac{1}{\delta}\rho^2\right)$$

and therefore

$$u \geq \frac{C(\delta)}{4\delta}\rho,$$

as stated in the proposition.

Lemmas 2.1, 2.2 and Proposition 2.3 immediately give us property (4) of §1. That is,

Theorem 2.5. *Let M be a compact Riemannian manifold with $\partial M \neq \emptyset$, satisfying an interior rolling* R*-ball condition and having sectional curvature $-a^2 \leq Sec(M) \leq b^2$ with $b \geq a$. Let $\rho(x) = d(x, \partial M)$ and u be the first eigenfunction satisfying Dirichlet conditions. Then on $M \setminus M_\delta$, with δ chosen as in Lemma 2.1, there are constants $C_i(a, b, H, R, D, n)$, $i = 1, 2$, such that*

$$C_1\rho(x) \leq u(x) \leq C_2\rho(x).$$

Furthermore, there exists $C = C(a, b, H, R, D, n) > 0$ such that $u(x) \leq Cu(y)$ for all $x, y \in M$ with $0 < d(x, \partial M) \leq 2d(y, \partial M)$.

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